Exam Lie Groups in Physics

Date	November 4, 2020
Time	8:30 - 10:30
Lecturer	D. Boer

- Write your name and student number on every separate sheet of paper
- All subquestions (a, b, etc) of the three exercises have equal weight
- Illegible handwriting will be graded as incorrect
- Good luck!

Problem 1

(a) Consider the sets of complex numbers C and nonzero complex numbers C$\backslash\{0\}$. Indicate the composition laws under which these sets form Lie groups. Explain your answers and explain why subtraction cannot be a composition law.

(b) Consider the group $SU(1,1)$ formed by the complex 2×2 matrices U with determinant equal to 1 that satisfy

$$U^\dagger = gU^{-1}g^{-1} \quad \text{with} \quad g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

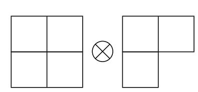
Give a parametrization of all the elements of $SU(1,1)$ and show that the set of those matrices indeed form a group under matrix multiplication.

(c) Explain what are the difference between $SU(1,1)$ and $SU(2)$ in terms of global properties (compact/connected/simply connected) and local properties (Killing form/Cartan matrix).

Problem 2

Consider the lie algebra $su(n)$ of the Lie group $SU(n)$ of unitary $n\times n$ matrices with determinant equal to 1.

(a) Decompose the following direct product of irreps of the Lie algebra $su(n)$



into a direct sum of irreps of $su(n)$, in other words, determine its Clebsch-Gordan series.

(b) Write down the dimensions of the irreps appearing in the obtained decomposition for $su(n)$ with n equal to the first digit of your student number plus 1. For example, if your student number is s2345678, then $n = 3$). Indicate inequivalent and complex conjugate irreps whenever appropriate.

(c) Write down the direct product between the defining representation $\boldsymbol{n}$ of $su(n)$ and its complex conjugate $\boldsymbol{n}^*$ in terms of Young tableaux and decompose it into irreps of $su(n)$. Use this decomposition to show that the adjoint representation is real.

Problem 3

Consider the Lie group $O(4)$ of orthogonal transformations in 4-dimensional Euclidean space and its subgroup $SO(4)$ of elements with determinant equal to 1.

(a) Determine the dimension of $O(4)$, by considering the constraints that need to be satisfied by its generators M_{ab} in the defining representation $(a, b = 1, 2, 3, 4)$.

The generators span the Lie algebra $o(4)$ of $O(4)$. The generators defined as $L_i = \frac{1}{2}\epsilon_{ijk}M_{jk}$ and $K_i = M_{i4}$ $(i, j, k = 1, 2, 3)$, satisfy the following algebra

$$[L_i, L_j] = i\epsilon_{ijk}L_k, \quad [L_i, K_j] = i\epsilon_{ijk}K_k, \quad [K_i, K_j] = i\epsilon_{ijk}L_k.$$

(b) Show explicitly that $o(4) \cong so(4)$ and $so(4) \cong so(3) \oplus so(3)$.

(c) Write down the expressions for the two Casimir operators of $o(4)$ in terms of the generators.

(d) Describe the general use of Casimir operators.